Episode 18

Inertial Properties of Rigid Bodies

ENGN0040: Dynamics and Vibrations

Allan Bower, Yue Qi

School of Engineering
Brown University

Topics for todays class

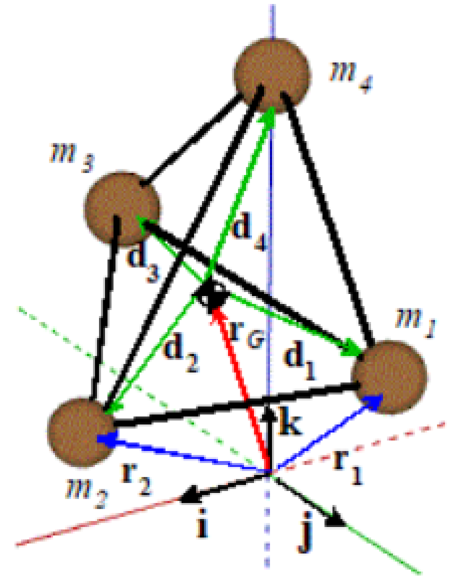
Inertial properties of rigid bodies

1. Definition of the inertia matrix
2. Angular momentum and KE in terms of the inertia matrix
3. How to calculate the inertia matrix
4. Changes to the inertia matrix caused by rotating a body
5. Understanding the inertia matrix
6. Simplified formulas for 2D problems
7. The parallel axis theorem
8. Proofs of all the formulas (optional!)

6.4 Inertial Properties of Rigid Bodies

Goal: Find formulas for ang momentum and KE in terms of $\underline{\omega}$

6.4.1 The Inertia Matrix



Rigid body will be idealized as infinite # particles connected by rigid links

For a system of particles

Total mass $M = \sum m_i$

Position of COM $\underline{r}_G = \frac{1}{M} \sum m_i \underline{r}_i$

Position relative to COM $\underline{d}_i = \underline{r}_i - \underline{r}_G$

Useful linear algebra formula

For a vector $\underline{u} = u_x \underline{i} + u_y \underline{j} + u_z \underline{k}$

$$\overbrace{\underline{u}\cdot\underline{u}}^{u_x^2+u_y^2+u_z^2}\,\underbrace{\underline{1}}_{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} - \underbrace{\underline{u}\otimes\underline{u}}_{\begin{bmatrix}u_x^2 & u_xu_y & u_xu_z\\ u_yu_x & u_y^2 & u_yu_z\\ u_zu_x & u_zu_y & u_z^2\end{bmatrix}} = \begin{bmatrix}u_y^2+u_z^2 & -u_xu_y & -u_xu_z\\ -u_yu_x & u_x^2+u_z^2 & -u_yu_z\\ -u_zu_x & -u_zu_y & u_x^2+u_y^2\end{bmatrix}$$

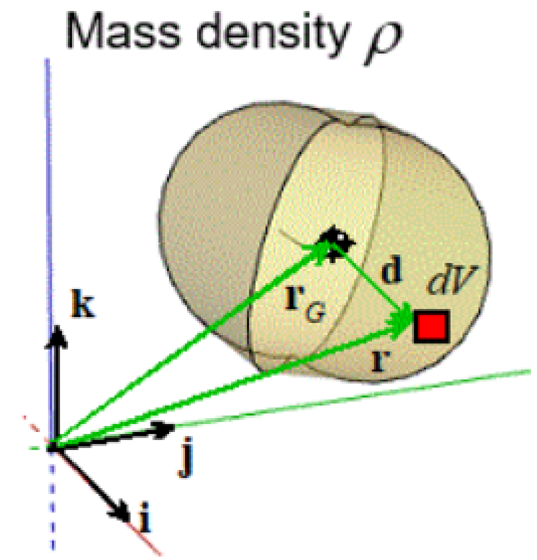
Mass moment of inertia

$$I_O = \sum_{\text{particles}} m_i\left[(\underline{r}_i\cdot\underline{r}_i)\underline{1} - \underline{r}_i\otimes\underline{r}_i\right] \quad (\text{about } O)$$

$$I_G = \sum_{\text{particles}} m_i\left[(\underline{d}_i\cdot\underline{d}_i)\underline{1} - \underline{d}_i\otimes\underline{d}_i\right] \quad (\text{about com})$$

I is always a 3×3 matrix $\begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{xy} & I_{yy} & I_{yz}\\ I_{xz} & I_{yz} & I_{zz}\end{bmatrix}$

Extend to rigid body



$m_i = \rho\, dV \qquad \sum_i m_i = \int_V \rho\, dV$

Total mass $M = \int_V \rho\, dV$

Center of mass $\underline{r}_G = \frac{1}{M} \int_V \underline{r}\, \rho\, dV$

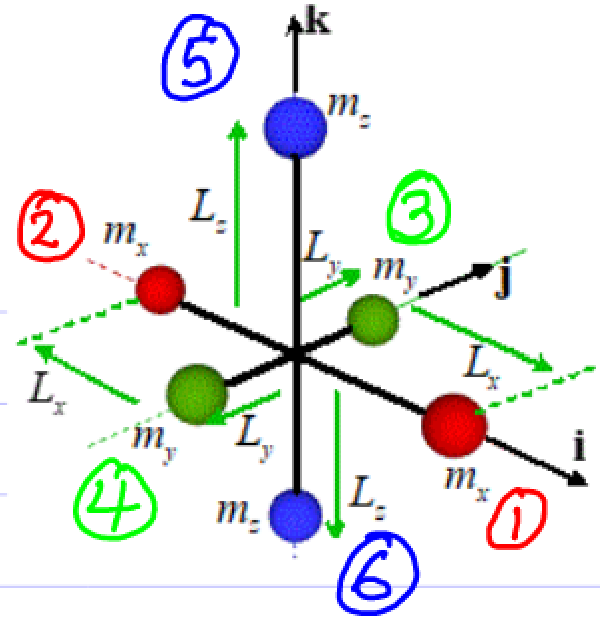
Define $\underline{d} = \underline{r} - \underline{r}_G$

Mass moments of inertia

$$\boxed{I_O = \int_V [(\underline{r} \cdot \underline{r})\underline{1} - \underline{r} \otimes \underline{r}]\, \rho\, dV \quad \text{(about } O\text{)}}$$

$$\boxed{I_G = \int_V [(\underline{d} \cdot \underline{d})\underline{1} - \underline{d} \otimes \underline{d}]\, \rho\, dV \quad \text{(about COM)}}$$

6.4.2 Example: Find the mass moment of inertia matrix about the COM for the simple system of particles shown



COM is at origin $\Rightarrow I_o = I_G$

① $\underline{d} = L_x \underline{i}$ ② $\underline{d} = -L_x \underline{i}$

③ $\underline{d} = L_y \underline{j}$ ④ $\underline{d} = -L_y \underline{j}$

⑤ $\underline{d} = L_z \underline{k}$ ⑥ $\underline{d} = -L_z \underline{k}$

$$I_o = I_G = 2m_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & L_x^2 & 0 \\ 0 & 0 & L_x^2 \end{bmatrix} + 2m_y \begin{bmatrix} L_y^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_y^2 \end{bmatrix} + 2m_z \begin{bmatrix} L_z^2 & 0 & 0 \\ 0 & L_z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow I_o = I_G = 2 \begin{bmatrix} m_y L_y^2 + m_z L_z^2 & 0 & 0 \\ 0 & m_x L_x^2 + m_z L_z^2 & 0 \\ 0 & 0 & m_x L_x^2 + m_y L_y^2 \end{bmatrix}$$

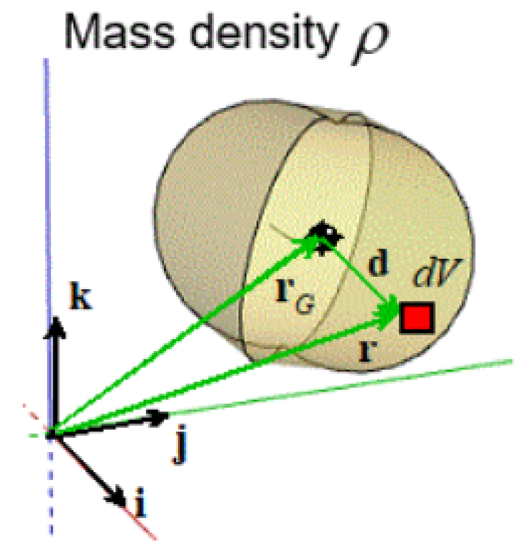
6.4.3 Angular Momentum and KE

For particles $\underline{h}_o = \sum_i m_i \underline{r}_i \times \underline{V}_i \qquad T = \sum \frac{1}{2} m_i |\underline{V}_i|^2$

For rigid body $\underline{h}_o = \int_V \underline{r} \times \underline{v}\, \rho\, dV \qquad T = \int_V \frac{1}{2} |\underline{V}_i|^2 \rho\, dV$

We can express these in terms of I_G

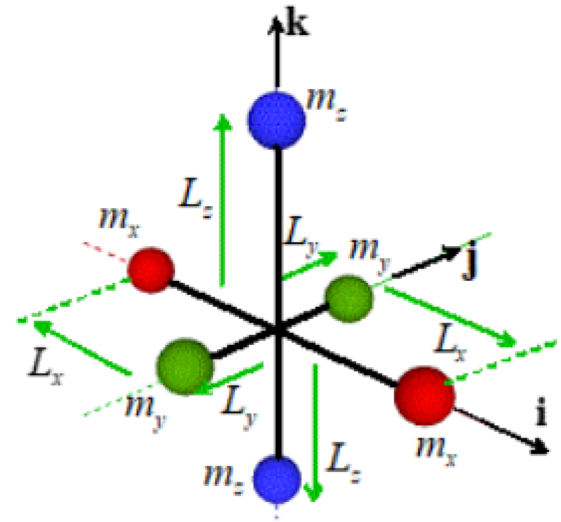
$$\boxed{\begin{aligned} \underline{h}_o &= \underbrace{\underline{r}_G \times M \underline{v}_G}_{\text{Translation (same as particle)}} + \underbrace{\underline{I}_G \underline{\omega}}_{\text{Rotation}} \\ T &= \overbrace{\tfrac{1}{2} M |\underline{V}_G|^2} + \overbrace{\tfrac{1}{2} \underline{\omega} \cdot \underline{I}_G \underline{\omega}} \end{aligned}}$$



6.4.4 Example: The system rotates about the **k** axis with angular speed ω_z. Find formulas for its angular momentum and kinetic energy:

(a) Using the formula in terms of the inertia matrix

(b) By summing the contributions from the individual particles directly



(a) Formula $\underline{h}_o = \underbrace{\cancel{\underline{r}_G}}_{=0} \times \underline{M}\, \underbrace{\cancel{\underline{v}_G}}_{=0} + \underline{I}_G\, \underline{\omega}$

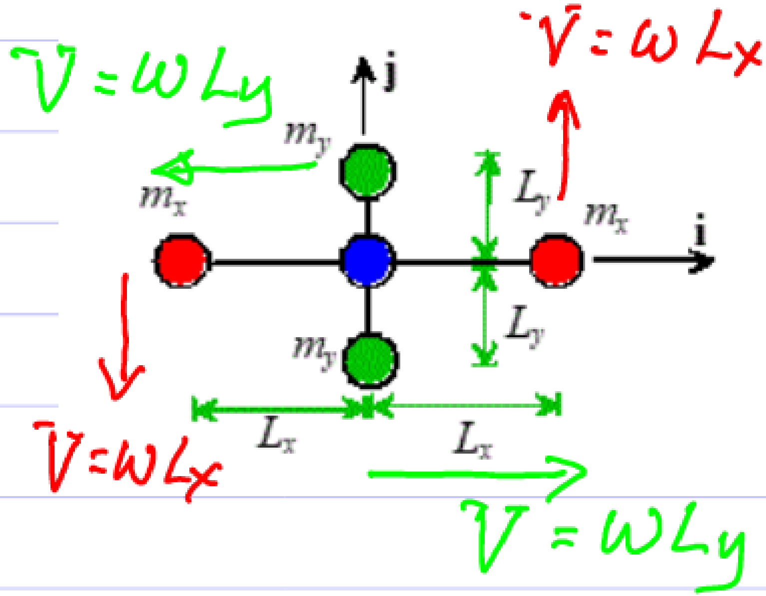
$$\Rightarrow \underline{h}_o = 2\begin{bmatrix} m_y L_y^2 + m_z L_z^2 & 0 & 0 \\ 0 & m_x L_x^2 + m_z L_z^2 & 0 \\ 0 & 0 & m_x L_x^2 + m_y L_y^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 2(m_x L_x^2 + m_y L_y^2)\,\omega \end{bmatrix} \Rightarrow \boxed{\underline{h}_o = 2(m_x L_x^2 + m_y L_y^2)\,\omega_z\, \underline{k}}$$

$$T = \frac{1}{2} M |\cancel{\underline{v}}_G|^2 + \frac{1}{2}\underline{\omega} \cdot I_G \underline{\omega} \quad (= 0) \quad \Rightarrow \boxed{T = (m_x L_x^2 + m_y L_y^2)\omega_z^2}$$

(b) By direct summation

Particles are in circular motion



$$\underline{h}_0 = \sum_i m_i \, \underline{r}_i \times \underline{V}_i$$

$$= 2(m_x L_x \cdot \omega L_x)\underline{k} + 2(m_y L_y \cdot \omega L_y)\underline{k}$$

$$\Rightarrow \underline{h}_0 = 2(m_x L_x^2 + m_y L_y^2)\,\omega\,\underline{k}$$

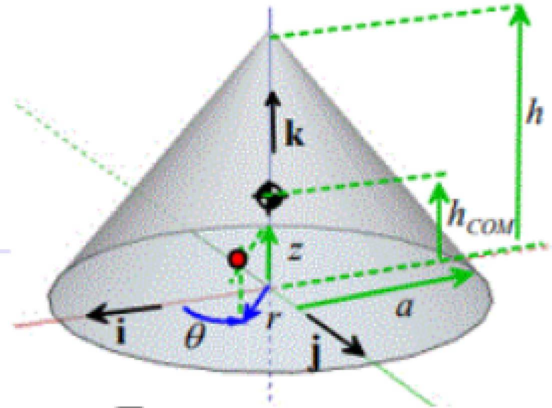
$$T = \sum_i \frac{1}{2} m_i |\underline{V}_i|^2 = \frac{1}{2}\left(2m_x(\omega L_x)^2 + 2m_y(\omega L_y)^2\right)$$

$$\Rightarrow \boxed{T = (m_x L_x^2 + m_y L_y^2)\,\omega^2}$$

6.4.5 Example: Find the mass moment of inertia matrix for the cone shown

(a) About the COM

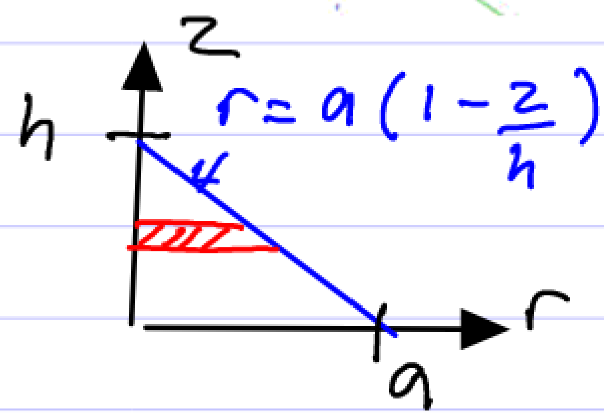
(b) About the origin



Do integral in polar coords

$$\underline{r} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + z\,\underline{k}$$

$$dV = r\,dr\,d\theta\,dz$$



$$M = \int_0^h \int_0^{2\pi} \int_0^{a(1-z/h)} \rho\, r\,dr\,d\theta\,dz$$

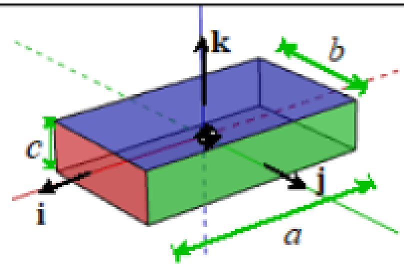
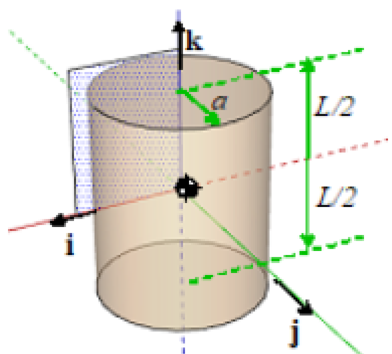
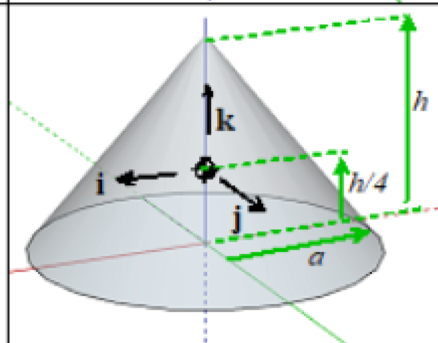
$$\underline{r}_G = \frac{1}{M}\int_V \underline{r}\,\rho\,dV \qquad \underline{d} = \underline{r} - \underline{r}_G$$

$$I_G = \int_V \left[\,|\underline{d}|^2\,\mathbb{1} - \underline{d}\otimes\underline{d}\,\right]\rho\,dV$$

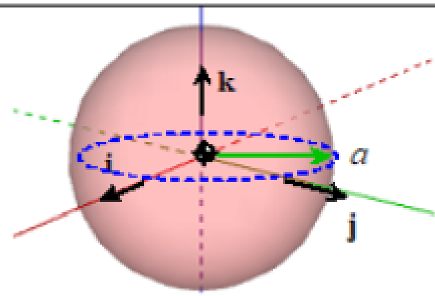
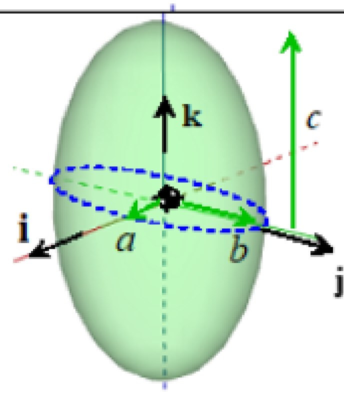
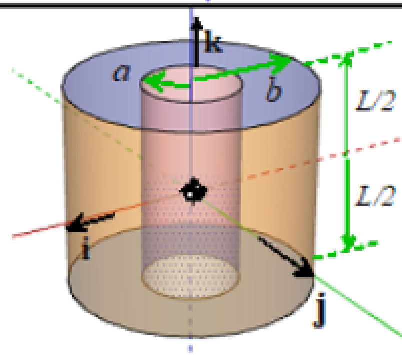
$$I_O = \int_V \left[\,|\underline{r}|^2\,\mathbb{1} - \underline{r}\otimes\underline{r}\,\right]\rho\,dV$$

Evaluate integrals with MATLAB

6.4.6 Inertia matrices for 3D bodies

Prism $M=\rho abc$		$$\frac{M}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$
Solid Cylinder $M=\pi\rho a^2 L$		$$\frac{ML^2}{12}\begin{bmatrix} 1+3a^2/L^2 & 0 & 0 \\ 0 & 1+3a^2/L^2 & 0 \\ 0 & 0 & 6a^2/L^2 \end{bmatrix}$$
Solid Cone $M=\frac{\pi}{3}\rho a^2 h$		$$\frac{3Ma^2}{20}\begin{bmatrix} 1+h^2/(4a^2) & 0 & 0 \\ 0 & 1+h^2/(4a^2) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Inertia matrices for 3D bodies

Solid Sphere $M=\frac{4}{3}\pi\rho a^3$		$\frac{2Ma^2}{5}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
Solid Ellipsoid $M=\frac{4}{3}\pi\rho abc$		$\frac{M}{5}\begin{bmatrix}b^2+c^2&0&0\\0&a^2+c^2&0\\0&0&a^2+b^2\end{bmatrix}$
Hollow Cylinder $M=\pi\rho(b^2-a^2)L$		$\frac{M}{12}\begin{bmatrix}L^2+3(a^2+b^2)&0&0\\0&L^2+3(a^2+b^2)&0\\0&0&6(a^2+b^2)\end{bmatrix}$

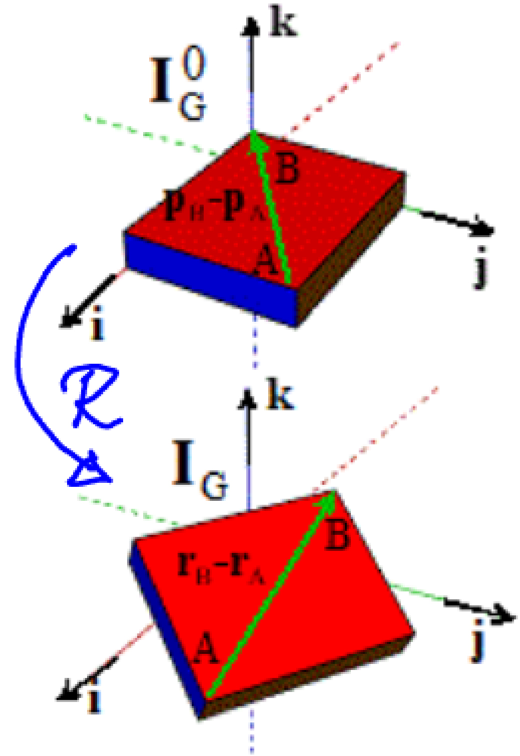
6.4.7 Changes to inertia matrix after rotation

Let: R = Rotation matrix

$W = (dR/dt)R^T$ = spin matrix

I_G^0 = inertia before rotation

I_G = inertia after rotation



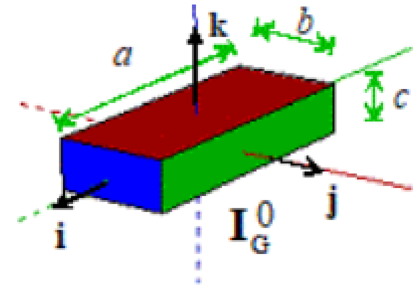
Then $\boxed{I_G = R\, I_G^0\, R^T}$

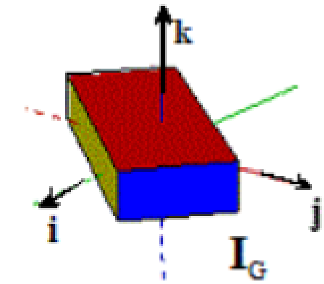
Time Derivative of I_G

$$\frac{dI_G}{dt} = \frac{dR}{dt} I_G^0 R^T + R I_G^0 \frac{dR^T}{dt}$$

$$\boxed{\frac{dI_G}{dt} = W I_G - I_G W}$$

Example: The prism rotates with angular speed ω_z about the **k** axis. Find
(a) The time derivative of the inertia matrix
(b) The inertia matrix after a 45 degree rotation





Use Formulas From tables

$$I_G^0 = \frac{m}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

Spin matrix (see L16)

$$W = \begin{bmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rotation Matrix

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

$$R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\frac{dI_G}{dt} = W I_G - I_G W \qquad I_G = R I_G^0 R^T$$

Use MATLAB

6.4.8 Understanding the moment of inertia

(1) I_G is the rotational equivalent of mass

Linear Momentum : $\underline{p} = M \underline{v}_G$

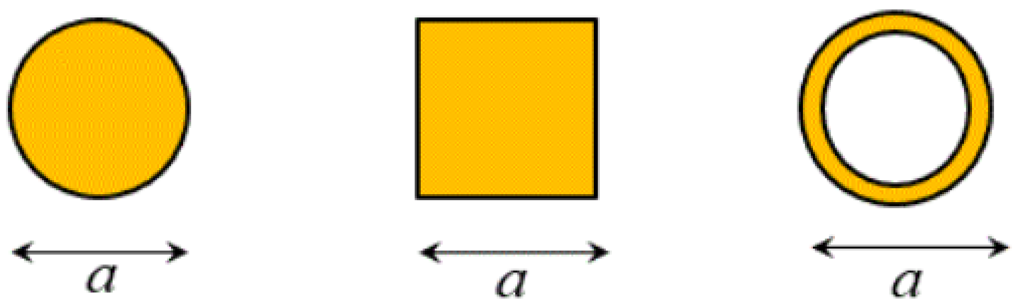
Rotational momentum : $\underline{h}^{rot} = I_G \underline{\omega}$

M : Translational inertia

I_G : Rotational inertia

(2) I_G quantifies mass distribution about COM

mass far from COM $\Rightarrow$ large I_G



$I_{Gzz} = \frac{M}{8} a^2$ $\qquad$ $I_{Gzz} = \frac{M}{6} a^2$ $\qquad$ $I_{Gzz} = \frac{M}{4} a^2$

(3) Inertia matrix is often diagonal

$$\underline{I}_G = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \qquad \underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

Then $\underline{h}^{rot} = I_{xx}\omega_x \underline{i} + I_{yy}\omega_y \underline{j} + I_{zz}\omega_z \underline{k}$

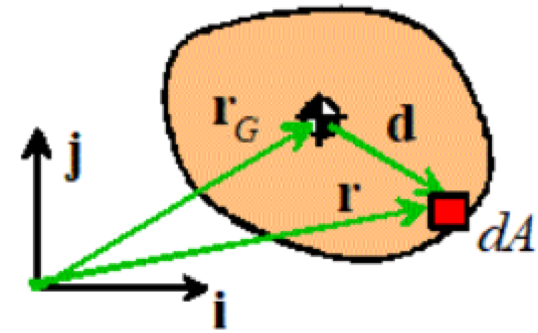
$\{I_{xx}, I_{yy}, I_{zz}\}$ specify inertia for rotation about $\underline{i}, \underline{j}, \underline{k}$ axes

(4) Inertia of a body can change with time (unlike mass)

$\Rightarrow$ (a) $\underline{h} = \text{const} \not\Leftrightarrow \underline{\omega} = \text{const}$

(b) A deformable body can change its orientation or angular velocity with no external force

6.4.9 Simplified formulas for $\underline{h}$ and $\underline{T}$ for 2D problems



μ = mass per unit area

For 2D solid I_O, I_G have form

$$I = \begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

zero by symmetry

Angular velocity $\underline{\omega} = \omega_z \underline{k}$

Hence $\boxed{I\underline{\omega} = I_{zz}\,\omega_z \underline{k} \qquad \underline{\omega}\cdot I\underline{\omega} = I_{zz}\,\omega_z^2}$

$\Rightarrow$ for 2D problems we only need I_{zz}

$$I_{Gzz} = \int_A (d_x^2 + d_y^2)\,\mu\, dA \qquad I_{Ozz} = \int_A (x^2+y^2)\,\mu\, dA$$

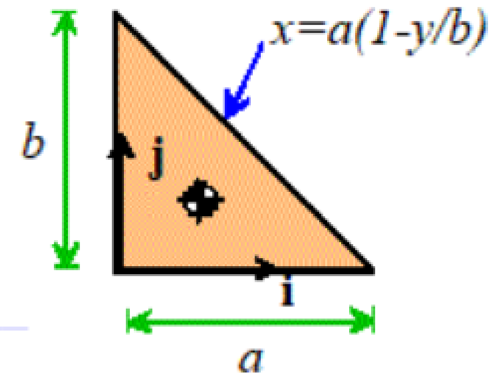
Note: I_{Gzz} & I_{Ozz} do not change if body is rotated about $\underline{k}$ axis ☺

2D formulas

$$\boxed{\begin{aligned} \underline{h}_O &= \underline{r}_G \times M\, \underline{v}_G + I_{Gzz}\, \omega_z\, \underline{k} \\ T &= \frac{1}{2} M\, |\underline{v}_G|^2 + \frac{1}{2} I_{Gzz}\, \omega_z^2 \end{aligned}}$$

6.4.10 Example: Find the mass moment of inertia I_{Gzz} for the triangular plate about the COM

Mass per unit area μ



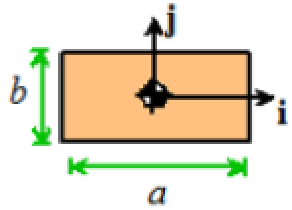
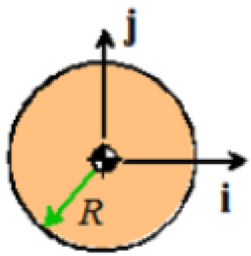
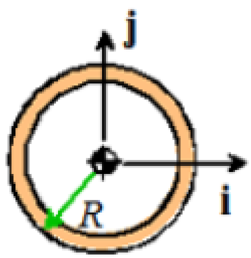
Mass $M = \int_0^b \int_0^{a(1-y/b)} \mu \, dx\, dy$

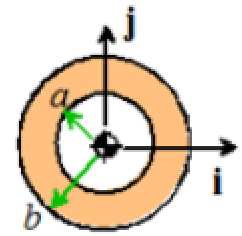
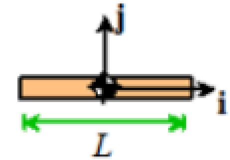
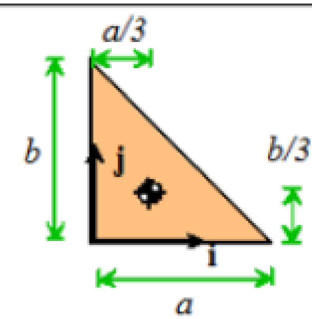
COM $\underline{r}_G = \frac{1}{M} \int_A \underline{r}\, \mu \, dA$ $\qquad \underline{d} = \underline{r} - \underline{r}_G$

$$I_{Gzz} = \int_A (d_x^2 + d_y^2)\, \mu \, dA$$

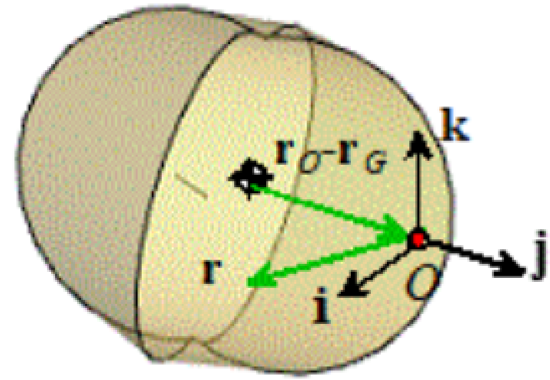
Evaluate integral with MATLAB

6.4.11 Moments of inertia for 2D bodies

Square		$I_{Gzz} = \frac{M}{12}(a^2 + b^2)$
Disk		$I_{Gzz} = \frac{M}{2} R^2$
Thin ring		$I_{Gzz} = MR^2$

Hollow disk		$I_{Gzz} = \frac{M}{2}\left(a^2 + b^2\right)$
Slender rod		$I_{Gzz} = \frac{M}{12} L^2$
Triangular Plate		$\frac{M}{18}(a^2 + b^2)$

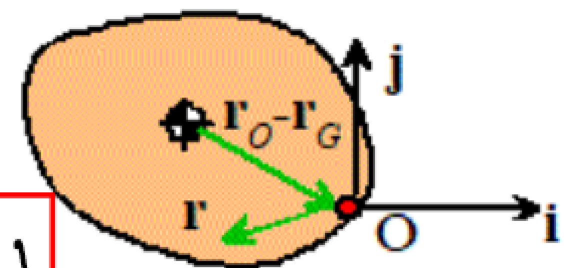
6.4.12 The Parallel Axis Theorem



Problem:

Given I_G, M, $\underline{r}_O$, $\underline{r}_G$

Find I_O



Formulas

Let $\underline{b} = \underline{r}_O - \underline{r}_G$

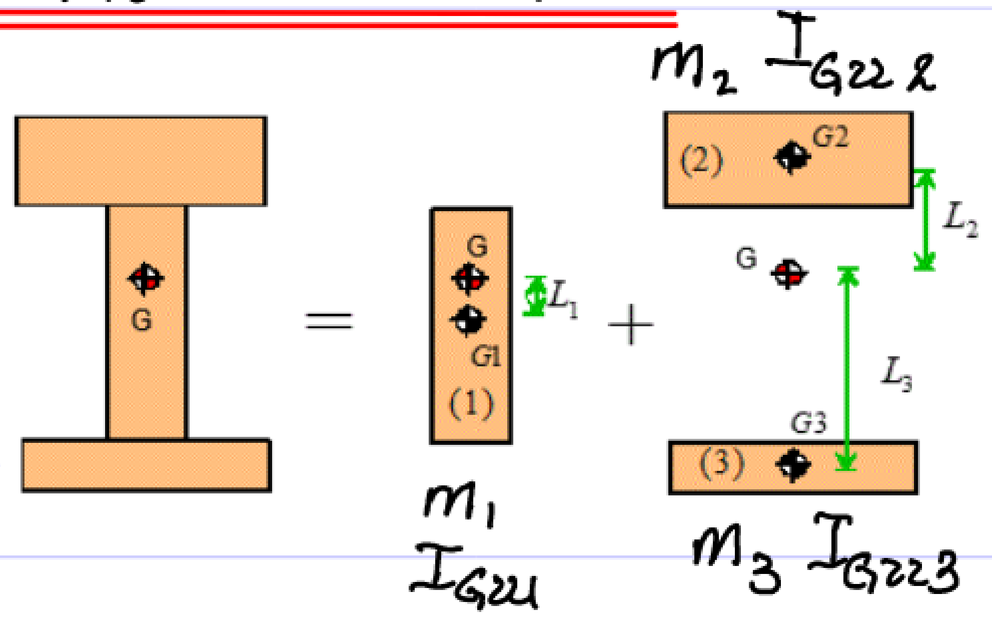
$$\boxed{I_O = I_G + M[(\underline{b}\cdot\underline{b})\mathbf{1} - \underline{b}\otimes\underline{b}] \qquad (3D)}$$

Let $L = |\underline{r}_O - \underline{r}_G|$ (distance from O to G)

$$\boxed{I_{Ozz} = I_{Gzz} + M L^2 \qquad (2D)}$$

6.4.13 Calculating I_G for connected bodies

To find I_G for complex shape:



(1) Find total mass $M = \sum_i m_i$

(2) Find combined COM

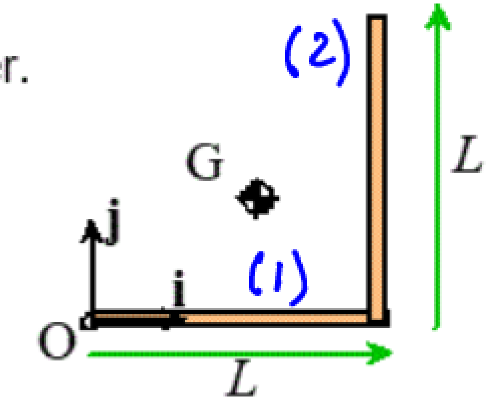
$$\underline{r}_G = \frac{1}{M} \sum_i m_i \underline{r}_{Gi}$$

③ Use parallel axis theorem to find I_G of each part about new COM and add

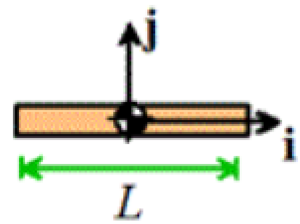
$$I_{Gzz} = I_{Gzz1} + m_1 L_1^2 + I_{Gzz2} + m_2 L_2^2 + I_{Gzz3} + m_3 L_3^2$$

(can also use (3) to find I_{Ozz})

Example 6.4.14: The rods each have mass m and are welded together.
Find
(1) $\mathbf{r}_G$
(2) I_{Ozz}
(3) I_{Gzz}



(From tables **Slender rod**



$I_{Gzz} = \frac{M}{12}L^2$

Total Mass $M = 2m$

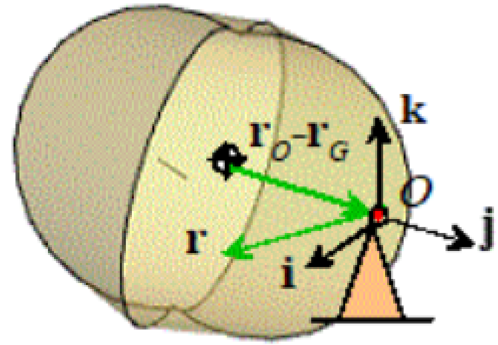
COM : $\underline{r}_G = \frac{1}{2m}\left\{ m\frac{L}{2}\underline{i} + m\left(L\underline{i} + \frac{L}{2}\underline{j}\right)\right\} = \frac{3L}{4}\underline{i} + \frac{L}{4}\underline{j}$

$$I_{Ozz} = \underbrace{\frac{mL^2}{12} + m\left(\frac{L}{2}\right)^2}_{\text{Bar (1)}} + \underbrace{\frac{mL^2}{12} + m\left(L^2 + \left(\frac{L}{2}\right)^2\right)}_{\text{Bar (2)}} = \frac{5}{3}mL^2$$

Find I_{Gzz} with parallel axis theorem

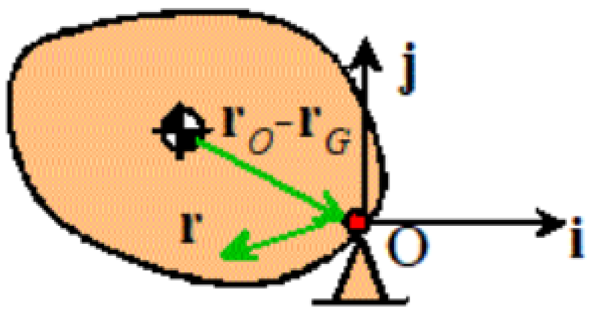
$$I_{Ozz} = I_{Gzz} + 2m\left\{\left(\frac{3L}{4}\right)^2 + \left(\frac{L}{4}\right)^2\right\} \Rightarrow I_{Gzz} = \frac{5}{12}mL^2$$

6.4.15 Angular momentum and KE of a body rotating about a stationary point



Method A: Use general formulas (Sects 6.4.3 & 6.4.9)

Method B (quicker)



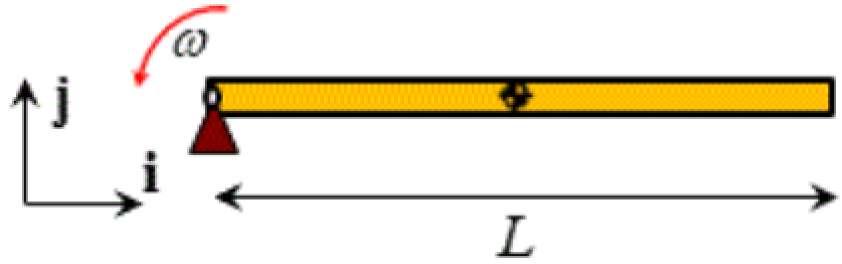
(1) Find I_0 (parallel axis thm)

(2) Use

$$\left.\begin{aligned} \underline{h} &= I_0\,\underline{\omega} \\ T &= \tfrac{1}{2}\,\underline{\omega}\cdot I_0\,\underline{\omega} \end{aligned}\right\} \text{3D}$$

$$\left.\begin{aligned} \underline{h} &= I_{0zz}\,\omega_z\,\underline{k} \\ T &= \tfrac{1}{2} I_{0zz}\,\omega_z^2 \end{aligned}\right\} \text{2D}$$

Example: Find the angular momentum and kinetic energy of the bar



Method A:

$$\underline{V}_G = \frac{L}{2}\omega \underline{j} \quad \text{(circular motion)}$$

$$\underline{h} = \underline{r}_G \times m\underline{V}_G + I_{Gzz}\omega_z \underline{k} = \frac{L}{2}\underline{i} \times m\frac{L}{2}\omega\underline{j} + \frac{mL^2}{12}\omega\underline{k}$$

$$\Rightarrow \underline{h} = \frac{mL^2}{3}\omega\underline{k}$$

$$T = \frac{1}{2}m|\underline{V}_G|^2 + \frac{1}{2}I_{Gzz}\omega_z^2 = \frac{1}{2}m\left(\frac{L}{2}\omega\right)^2 + \frac{mL^2}{24}\omega_z^2$$

$$\Rightarrow T = (mL^2/6)\,\omega_z^2$$

Method B: $I_O = mL^2/12 + m(L/2)^2 = mL^2/3$

$$\Rightarrow \underline{h} = \frac{mL^2}{3}\omega_z\underline{k} \qquad T = \frac{1}{2}\frac{mL^2}{3}\omega_z^2$$

Deriving the inertia formulas

(1) Prove the angular momentum formula $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G \boldsymbol{\omega}$

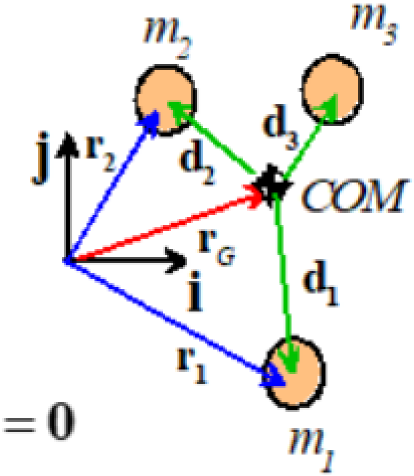
(2) Prove the kinetic energy formula $T = \frac{1}{2} M |\mathbf{v}_G|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}$

(3) Prove the parallel axis theorem $\mathbf{I}_O = \mathbf{I}_G + M\left[(\mathbf{b} \cdot \mathbf{b})\mathbf{1} - \mathbf{b} \otimes \mathbf{b}\right]$

(4) Prove the rotation formula $\mathbf{I}_0 = \mathbf{R}\mathbf{I}_0^0 \mathbf{R}^T$

(5) Derive the time derivative of the inertia matrix $\frac{d\mathbf{I}_0}{dt} = \mathbf{W}\mathbf{I}_0 - \mathbf{I}_0 \mathbf{W}$

Review of some previous results



Total mass $M = \sum_{Particles} m_i$

Position of COM (definition) $\mathbf{r}_G = \frac{1}{M} \sum_{Particles} m_i \mathbf{r}_i$

Property of COM (see L10 for proof) $\mathbf{d}_i = \mathbf{r}_i - \mathbf{r}_G \Rightarrow \sum_{Particles} m_i \mathbf{d}_i = \mathbf{0}$

For rigid body $\int_V \mathbf{d} \rho dV = \mathbf{0}$

Angular momentum of a system of particles (see L10 for proof)

$$\mathbf{h}^{TOT} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} \mathbf{d}_i \times m_i(\mathbf{v}_i - \mathbf{v}_G)$$

Rotation and spin matrices $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{1}$ $\quad \mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T$ $\quad \mathbf{W}^T = -\mathbf{W}$

Rigid body kinematics formula $\mathbf{v}_i - \mathbf{v}_G = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_G)$

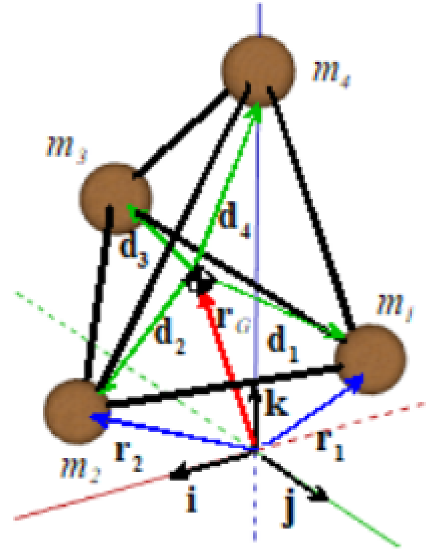
Proof of the angular momentum formula

Prove $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}$ with $\mathbf{I}_G = \sum_{Particles} m_i \left[(\mathbf{d}_i \cdot \mathbf{d}_i)\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_i \right]$

(1) Angular Momentum $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} \mathbf{d}_i \times m_i(\mathbf{v}_i - \mathbf{v}_G)$

(2) Rigid body formula $\mathbf{v}_i - \mathbf{v}_G = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_G) = \boldsymbol{\omega} \times \mathbf{d}_i$

(3) Combine $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} m_i \mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i$



(4) Triple cross product formula $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$$\Rightarrow \mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i = (\mathbf{d}_i \cdot \mathbf{d}_i)\boldsymbol{\omega} - \mathbf{d}_i(\mathbf{d}_i \cdot \boldsymbol{\omega})$$

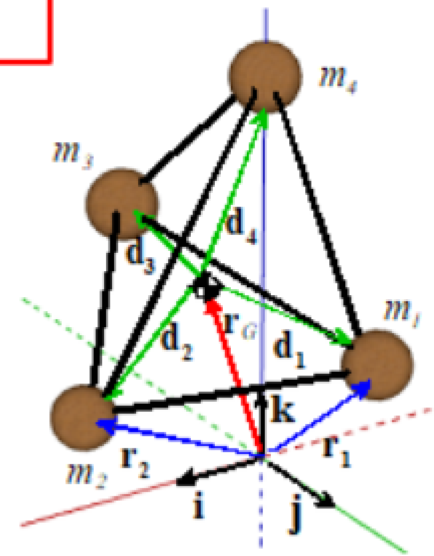
(5) Factor $(\mathbf{d}_i \cdot \mathbf{d}_i)\boldsymbol{\omega} - \mathbf{d}_i(\mathbf{d}_i \cdot \boldsymbol{\omega}) = \left[(\mathbf{d}_i \cdot \mathbf{d}_i)\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_i \right]\boldsymbol{\omega}$

Hence $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} m_i \left[(\mathbf{d}_i \cdot \mathbf{d}_i)\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_i \right]\boldsymbol{\omega}$

$$\boxed{\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}}$$

Proof of the kinetic energy formula

Prove $T = \frac{1}{2} M \left| \mathbf{v}_G \right|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}$ with $\mathbf{I}_G = \sum_{Particles} m_i \left[(\mathbf{d}_i \cdot \mathbf{d}_i)\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_i \right]$



By definition $T = \sum_{particles} \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i$

Rigid body formula $\mathbf{v}_i - \mathbf{v}_G = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_G) \Rightarrow \mathbf{v}_i = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{d}_i$

Combine $T = \sum_{particles} \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_{particles} \frac{1}{2} m_i (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{d}_i) \cdot (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{d}_i)$

Expand $= (\mathbf{v}_G \cdot \mathbf{v}_G) \frac{1}{2} \underbrace{\boxed{\sum_{particles} m_i}}_{= M} + \mathbf{v}_G \cdot \left(\boldsymbol{\omega} \times \underbrace{\boxed{\sum_{particles} m_i \mathbf{d}_i}}_{\boxed{= 0}} \right) + \frac{1}{2} \sum_{particles} m_i (\boldsymbol{\omega} \times \mathbf{d}_i) \cdot (\boldsymbol{\omega} \times \mathbf{d}_i)$

Recall vector formula $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$

$$\Rightarrow (\boldsymbol{\omega} \times \mathbf{d}_i) \cdot (\boldsymbol{\omega} \times \mathbf{d}_i) = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{d}_i \cdot \mathbf{d}_i) - (\boldsymbol{\omega} \cdot \mathbf{d}_i)^2$$

Factor $(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{d}_i \cdot \mathbf{d}_i) - (\boldsymbol{\omega} \cdot \mathbf{d}_i)^2 = \boldsymbol{\omega} \cdot \left[(\mathbf{d}_i \cdot \mathbf{d}_i)\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_i \right] \boldsymbol{\omega}$

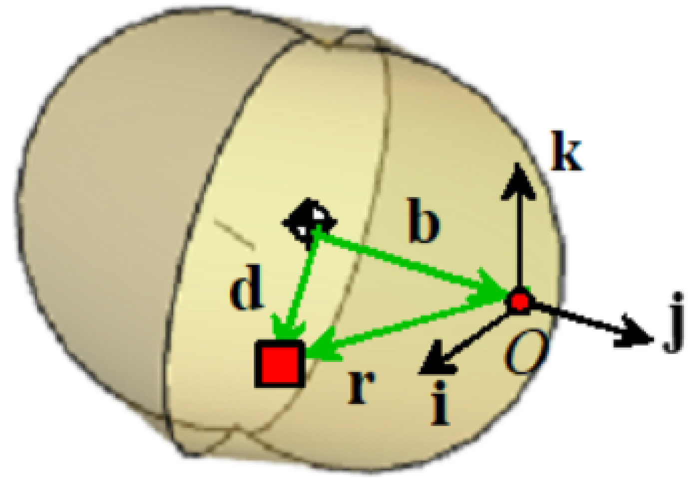
Hence $T = \frac{1}{2} M \mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2} \sum_{particles} \boldsymbol{\omega} \cdot \left[(\mathbf{d}_i \cdot \mathbf{d}_i)\mathbf{1} - \mathbf{d}_i \otimes \mathbf{d}_i \right] \boldsymbol{\omega} \Rightarrow \boxed{T = \frac{1}{2} M \mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}}$

Proof of the parallel axis theorem

Prove $\mathbf{I}_O = \mathbf{I}_G + M\left[(\mathbf{b}\cdot\mathbf{b})\mathbf{1} - \mathbf{b}\otimes\mathbf{b}\right]$

$$\mathbf{I}_G = \int_V \left[(\mathbf{d}\cdot\mathbf{d})\mathbf{1} - \mathbf{d}\otimes\mathbf{d}\right]\rho dV$$

$$\mathbf{I}_O = \int_V \left[(\mathbf{r}\cdot\mathbf{r})\mathbf{1} - \mathbf{r}\otimes\mathbf{r}\right]\rho dV$$



Geometry $\mathbf{r} = \mathbf{b} - \mathbf{d}$

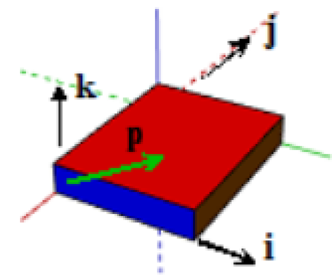
Substitute $\mathbf{I}_O = \int_V \Big[\big([\mathbf{b}-\mathbf{d}]\cdot[\mathbf{b}-\mathbf{d}]\big)\mathbf{1} - [\mathbf{b}-\mathbf{d}]\otimes[\mathbf{b}-\mathbf{d}]\Big]\rho dV$

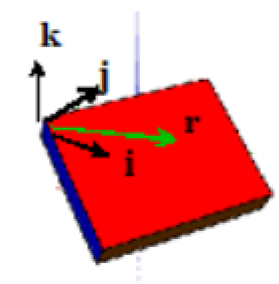
Expand $\mathbf{I}_O = \underbrace{\boxed{\int_V \left[(\mathbf{d}\cdot\mathbf{d})\mathbf{1} - \mathbf{d}\otimes\mathbf{d}\right]\rho dV}}_{=\mathbf{I}_G} + \underbrace{\boxed{\int_V \rho dV}}_{=M}\left[(\mathbf{b}\cdot\mathbf{b})\mathbf{1} - \mathbf{b}\otimes\mathbf{b}\right] - 2\left(\mathbf{b}\cdot\underbrace{\boxed{\int_V \mathbf{d}\rho dV}}_{=0}\right)\mathbf{1} + \mathbf{b}\otimes\underbrace{\boxed{\int_V \mathbf{d}\rho dV}}_{=0} + \underbrace{\boxed{\int_V \mathbf{d}\rho dV}}_{=0}\otimes\mathbf{b}$

$$\boxed{\mathbf{I}_O = \mathbf{I}_G + M\left[(\mathbf{b}\cdot\mathbf{b})\mathbf{1} - \mathbf{b}\otimes\mathbf{b}\right]}$$

Proof of the inertia rotation formula

$\boxed{\text{Prove } \mathbf{I}_0 = \mathbf{R}\mathbf{I}_0^0\mathbf{R}^T \quad \text{with} \quad \mathbf{I}_0^0 = \int_V [(\mathbf{p}\cdot\mathbf{p})\mathbf{1} - \mathbf{p}\otimes\mathbf{p}]\,\rho dV, \quad \mathbf{I}_O = \int_V [(\mathbf{r}\cdot\mathbf{r})\mathbf{1} - \mathbf{r}\otimes\mathbf{r}]\,\rho dV}$





(1) Rotation formula $\mathbf{r} = \mathbf{R}\mathbf{p}$

(2) Substitute $\mathbf{I}_O = \int_V [(\mathbf{R}\mathbf{p}\cdot\mathbf{R}\mathbf{p})\mathbf{1} - \mathbf{R}\mathbf{p}\otimes\mathbf{R}\mathbf{p}]\,\rho dV$

(3) Linear Algebra Formulas $(\mathbf{A}\mathbf{a})\cdot(\mathbf{B}\mathbf{b}) = \mathbf{a}\cdot(\mathbf{A}^T\mathbf{B})\mathbf{b}$ $\quad \mathbf{A}\mathbf{a}\otimes\mathbf{B}\mathbf{b} = \mathbf{A}(\mathbf{a}\otimes\mathbf{b})\mathbf{B}^T$

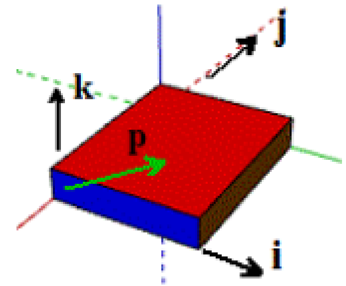
(4) Orthogonality of **R** $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{1}$

(5) Rearrange integrand in (2) $[(\mathbf{R}\mathbf{p}\cdot\mathbf{R}\mathbf{p})\mathbf{1} - \mathbf{R}\mathbf{p}\otimes\mathbf{R}\mathbf{p}] = [(\mathbf{p}\cdot\mathbf{R}^T\mathbf{R}\mathbf{p})\mathbf{R}\mathbf{R}^T - \mathbf{R}\mathbf{p}\otimes\mathbf{p}\mathbf{R}^T]$

(6) Factor $\mathbf{I}_O = \mathbf{R}\left\{\int_V [(\mathbf{p}\cdot\mathbf{p})\mathbf{1} - \mathbf{p}\otimes\mathbf{p}]\,\rho dV\right\}\mathbf{R}^T$ $\quad \boxed{\Rightarrow \mathbf{I}_0 = \mathbf{R}\mathbf{I}_0^0\mathbf{R}^T}$

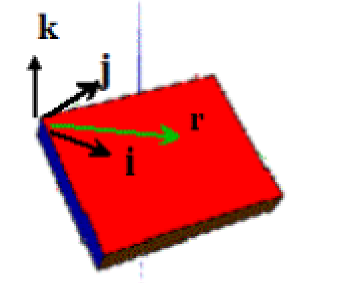
Time derivative of the inertia matrix

Prove $\dfrac{d\mathbf{I}_0}{dt} = \mathbf{W}\mathbf{I}_0 - \mathbf{I}_0\mathbf{W}$



(1) Rotation formula $\mathbf{I}_0 = \mathbf{R}\mathbf{I}_0^0\mathbf{R}^T$

(2) Time derivative $\dfrac{d\mathbf{I}_0}{dt} = \dfrac{d\mathbf{R}}{dt}\mathbf{I}_0^0\mathbf{R}^T + \mathbf{R}\mathbf{I}_0^0\dfrac{d\mathbf{R}^T}{dt}$



(3) From (1) $\mathbf{R}^T\mathbf{I}_0\mathbf{R} = \mathbf{I}_0^0 \Rightarrow \dfrac{d\mathbf{I}_0}{dt} = \underbrace{\boxed{\dfrac{d\mathbf{R}}{dt}\mathbf{R}^T}}_{=\mathbf{W}}\mathbf{I}_0\overbrace{\boxed{\mathbf{R}\mathbf{R}^T}}^{=1} + \overbrace{\boxed{\mathbf{R}\mathbf{R}^T}}^{=1}\mathbf{I}_0\underbrace{\boxed{\mathbf{R}\dfrac{d\mathbf{R}^T}{dt}}}_{=\mathbf{W}^T}$

(4) Recall $\mathbf{W}^T = -\mathbf{W}$

(5) Hence $\boxed{\dfrac{d\mathbf{I}_0}{dt} = \mathbf{W}\mathbf{I}_0 - \mathbf{I}_0\mathbf{W}}$